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PARABOLIC APPROXIMATION TO THE THEORY OF TRANSVERSE VIBRATIONS OF RODS AND BEAMS

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A parabolic equation is presented which describes the transverse vibrations of beams and which considers the elastic shear energy. This equation gives corrections to the frequencies of the classical Bernoulli–Euler theory which have the same accuracy as the corrections in the Timoshenko shaft model, but it does not have the additional non-physical vibration modes predicted by the Timoshenko model.

Up to now, classical Bernoulli–Euler theory [1] has been used for calculating transverse vibrations in rods and shafts, with only two notable generalizations. Rayleigh [2] considered the rotational inertia of separate elements of the beam as it vibrates, and Timoshenko's model made a substantial step forward at the turn of the century [3] by considering not only bending, but also shear strains during transverse vibrations. All attempts to obtain new approximations of this problem by starting from the exact equations of three-dimensional elastic theory lead to extremely large and complex structures (see [4] for example).

Opinion has strengthened that the success of Timoshenko's theory lies in the fact that his model is a hyperbolic approximation to the exact elastic equations [4]. However, we wish to show that Timoshenko's frequency corrections can be obtained by working with a parabolic equation that describes the transverse vibrations of a beam. We now present the basic moments of Timoshenko's theory [1, 3, 4]. Let $y_1(t, x)$ and $y_2(t, x)$ be the transverse beam displacements caused by bending and shear, respectively. Timoshenko's model specifies the following Lagrange functions:

$$L = T - V; \tag{1}$$

$$T = \frac{\rho F}{2}\int_0^l (\dot{y}_1 + \dot{y}_2)^2 dx + \frac{\rho I}{2}\int_0^l (\dot{y}_1')^2 dx; \tag{2}$$

$$V = \frac{EI}{2}\int_0^l (y_1'')^2 dx + \frac{kFG}{2}\int_0^l (y_2')^2 dx. \tag{3}$$

Here E is the elastic modulus of the beam material; G is the shear modulus; ρ is the bulk density; I is the moment of inertia of a transverse section of the beam relative to the axis which passes through the center of gravity of this section and which is perpendicular to the vibration plane; F is the transverse cross section of the shaft; l is its length; k is the shear coefficient; a dot denotes differentiation with respect to time t; and a prime denotes differentiation with respect to x.

The Lagrange functions (1)-(3) lead to equations of motion for $y = y_1 + y_2$ and $\psi = y' - y_2'$:

$$\ddot{y} - k\frac{G}{\rho}(y'' - \psi') = 0,\ \psi'' - \frac{\rho}{E}\ddot{\psi} + k\frac{GF}{EI}(y' - \psi) = 0, \tag{4}$$

and also to boundary conditions, which for a hinged shaft have the form

$$\begin{gathered} y(t, 0) = y(t, l) = 0,\ y''(t, 0) = y''(t, 0) = 0, \\ \psi'(t, 0) = \psi'(t, l) = 0. \end{gathered} \tag{5}$$

We now substitute into Eq. (4) the standard form to shaft vibrations

$$\begin{gathered} y(t, x) = A_n \sin(\omega_n t + \varepsilon_n)\sin(\lambda_n x), \\ \psi(t, x) = B_n \sin(\omega_n t + \varepsilon_n)\cos(\lambda_n x),\ \lambda_n = n\pi/l. \end{gathered} \tag{6}$$

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which satisfy the boundary conditions (5). As a result, we obtain the frequency equation

$$a^2\lambda_n^4 - \omega^2 - \left(1 + \frac{E}{kG}\right) r^2\lambda_n^2\omega^2 + r^2\frac{\rho}{kG}\omega^4 = 0, \tag{7}$$

where $a^2 = IE/F\rho$ and r is the radius of inertia of the transverse section of the beam: $r^2 = I/F$. There is also a relation between the vibration amplitudes A_n and B_n:

$$\frac{B_n}{A_n} = \lambda_n\left[1 - \frac{E}{kG}\frac{\omega_n^2}{\Omega_n^2}r^2\lambda_n^2\right]. \tag{8}$$

Here Ω_n is the beam vibration frequency from the classical Bernoulli–Euler theory [1], which considers only the first two terms in Eq. (7):

$$\Omega_n = a\lambda_n^2. \tag{9}$$

Equation (7) defines two series of frequencies, of which only the low-frequency vibration mode transforms directly to Bernoulli–Euler frequencies (9) when the small parameter $r^2\lambda_n^2$ in Timoshenko's theory [5] is set to zero. Timoshenko himself [1, 3] examined only the fundamental low-frequency vibration mode, and solved Eq. (7) by iteration, by considering that $r^2\lambda_n^2$ was small [6]. The effect of shear on the bending vibrations is considered by the term proportional to E/kG in Eq. (7). As a result, the following formula is obtained for the frequencies [1]:

$$\omega_n \simeq a\lambda_n^2\left[1 - \frac{1}{2}\left(1 + \frac{E}{kG}\right)r^2\lambda_n^2\right]. \tag{10}$$

As revealed in [5], the second high-frequency vibration mode plays an auxiliary role in Timoshenko's theory. Vibrations are hardly ever excited with these frequencies, but considering them in the frequency equation improves the fundamental vibration frequency.

Below we show that the same frequency corrections as in Eq. (10) can be found by using a parabolic equation which predicts Timoshenko's transverse beam vibrations, but without the non-physical frequency doubling. The basic idea of this approach is that we attempt to eliminate the variable y_2 from Eqs. (1)-(3) but still consider the effect of these strains on the bending vibrations.

The basic contribution to the first term in Eq. (2), which determines the kinetic energy of the beam, comes from the variable $\dot{y}_1$, to which $\dot{y}_2$ is only a correction. Actually, by using explicit solutions, e.g., for a hinged shaft (6), and by using the relationship (8) between the amplitudes, we obtain

$$\frac{\dot{y}_2}{\dot{y}_1} = \frac{r^2\lambda_n^2\frac{E}{kG}\left(\frac{\omega_n}{\Omega_n}\right)^2}{1 - r^2\lambda_n^2\frac{E}{kG}\left(\frac{\omega_n}{\Omega_n}\right)^2}. \tag{11}$$

For a sufficiently long beam (h/*l* is small, where h is the beam thickness), $\omega_n^2/\Omega_n^2 \sim 1$ (see [5] for example). As a result, Eq. (11) gives

$$\frac{\dot{y}_2}{\dot{y}_1} \simeq r^2\lambda_n^2 \ll 1. \tag{12}$$

It should be noted that this estimate is only for the fundamental vibration mode in Timoshenko's theory. Thus, the variable $y_2$can be eliminated in Eq. (1).

The second term in Eq. (3) represents the elastic shear energy. The variable y_2' in Eq. (3) can be expressed in terms of y_1. Actually, from Hooke's law, the shear strain is

$$y_2' = \frac{Q}{kGF},$$

where Q is the shearing force which arises during transverse beam vibrations. From elementary beam theory [1], we have

$$Q = -EI\frac{d^3y_1}{dx^3}.$$

Consequently,

$$y_2' = -\frac{EI}{kFG}y'''_1. \tag{13}$$

In order to determine the correct Lagrange function for bending vibrations, Eq. (13) must be substituted into Eq. (3) and the sign of the second term must be reversed. This accounts for the fact that part of the energy of bending vibrations goes into shear energy.

We finally obtain the Lagrange function in our model

$$L_1 = \frac{\rho F}{2}\int_0^l \dot{y}^2 dx + \frac{\rho I}{2}\int_0^l \dot{y}'^2 dx - \frac{EI}{2}\int_0^l y''^2 dx + \frac{E^2 I^2}{2kFG}\int_0^l y'''^2 dx. \tag{14}$$

The subscript has been omitted from the bending variable y(t, x) for simplicity.

The variation of Eq. (14) produces a new parabolic equation, which considers the effect of shear strains on transverse beam oscillations:

$$\ddot{y} + a^2 y_x^{(4)} - r^2 \ddot{y}'' + a^2 r^2 \frac{E}{kG} y_x^{(6)} = 0 \tag{15}$$

where $y_x^{(n)} \equiv \partial^n y/\partial x^n$. For a hinged shaft the variation of Eq. (14) must be performed for the following end conditions:

$$y(t, 0) = y(t, l) = 0,\ y''(t, 0) = y''(t, l) = 0. \tag{16}$$

By substituting the standard vibration form dictated by the boundary conditions (16) into Eq. (15), we find the frequency equation

$$\omega_n^2(1 + r^2\lambda_n^2) - a^2\lambda_n^4\left(1 - \frac{E}{kG}r^2\lambda_n^2\right) = 0. \tag{17}$$

When we solve this equation with to the same accuracy used for the frequency corrections in Timoshenko's model, we arrive at Eq. (10):

$$\omega_n = a\lambda_n^2\sqrt{\frac{1 - \frac{E}{kG}r^2\lambda_n^2}{1 + r^2\lambda_n^2}} \simeq a\lambda_n^2\left[1 - \frac{1}{2}\left(1 + \frac{E}{kG}\right)r^2\lambda_n^2\right].$$

Thus, our Eq. (15) gives the same corrections to the Bernoulli–Euler frequencies as Timoshenko's model. By using an explicit solution for y(t, x) from Eq. (6) and the frequency equation (17), it is easy to convince oneself that the energy in our model, which corresponds to the Lagrange function (14), is positive definite:

$$E_n = \frac{1}{4}A_n^2 EIl\lambda_n^4\left(1 - \frac{E}{kG}r^2\lambda_n^2\right).$$

For a rectangular beam, E/kG = 3.2 (see [1], for example). Within the region where this theory can be applied ($r^2\lambda_n^2 \ll 1$), E_n is positive.

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